

A topological approach to left eigenvalues of quaternionic matrices*

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October 11, 2012

Abstract

It is known that a 2×2 quaternionic matrix has one, two or an infinite number of left eigenvalues, but the available algebraic proofs are difficult to generalize to higher orders. In this paper a different point of view is adopted by computing the topological degree of a characteristic map associated to the matrix and discussing the rank of the differential. The same techniques are extended to 3×3 matrices, which are still lacking a complete classification.

Keywords Quaternionic matrices, Left eigenvalues, Characteristic map, Topological degree.

MSC 15A33, 15A18

1 INTRODUCTION

In 1985, Wood [25] proved that any $n \times n$ quaternionic matrix A has at least one *left eigenvalue*, that is a quaternion $\lambda \in \mathbb{H}$ such that the matrix $A - \lambda \text{Id}$ is singular. However, even for matrices of small size the left spectrum is not fully understood yet (see Zhang's papers [26, 27] for a survey). For instance, it was only in 2001 when Huang and So [13] proved that a 2×2 matrix may have one, two or an infinite number of left eigenvalues; a different proof was presented by the authors in [21]. While Wood used topological techniques, namely homotopy groups, the two latter papers are of algebraic nature and seemingly difficult to generalize for $n > 2$.

In this article we try a different approach. The basic ideas will be those of characteristic map, linearization and topological degree. In the simplest case $n = 2$ we associate to each matrix A a polynomial μ_A whose roots are the

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left eigenvalues; computing the rank of its differential allows us to classify the different types of spectra.

In the second part of the paper we extend those techniques to 3×3 matrices. This time the characteristic map may not be a polynomial but a rational map, usually with a point of discontinuity. This seems to require the use of a local version of degree, although a closer look allows us to reduce the problem to the global theory. In particular, this gives a new proof of the existence of left eigenvalues.

Unlike the 2×2 case, where the linear equations that appear correspond to the well known Sylvester equation, the 3×3 situation is much more complex. Then for $n = 3$ a complete classification of spectra is still unknown. Nevertheless our method allows to deal with specific examples and opens the way for a better understanding of the general case.

The paper is organized as follows. In Section 2 we recall some topological and algebraic preliminaries. Although our ideas are closely related to the theory of quasideterminants [10], we have preferred a development based on Study's determinant which parallels the commutative setting. Section 3 is devoted to a notion of *characteristic map* for the left eigenvalues of a quaternionic matrix A , that is a map $\mu_A: \mathbb{H} \rightarrow \mathbb{H}$ such that $\mu_A(\lambda) = 0$ if and only if the matrix $A - \lambda \text{Id}$ is not invertible. In Section 4 we give a complete classification of the left spectra of 2×2 matrices, depending on the rank of the characteristic map. In Section 5 we prove that any 3×3 matrix A has a characteristic map μ_A which is either a polynomial (when some of the entries outside the diagonal is null) or a rational function with a distinguished point π_A called its *pole*. When μ_A is continuous it has topological degree 3. However, π_A may be a point of discontinuity; in this case the matrix $B = A - \pi_A \text{Id}$ turns out to be invertible and we prove that B and B^{-1} have diffeomorphic spectra and that B^{-1} admits a polynomial characteristic map. The last Section offers several illustrative examples.

2 PRELIMINARIES

2.1 Topological degree

The topological degree (or Brouwer degree) of a map can be defined by techniques either from algebraic topology [5] or from functional analysis [1, 4, 23]. We want to apply the following global result (cf. [19, p. 101]):

Theorem 1. *Let M be a connected closed oriented manifold, let $\mu: M \rightarrow M$ be a differentiable map of degree k . Let $m \in M$ be a regular value such that the differential $\mu_{*\lambda}$ preserves the orientation for any λ in the fiber $\mu^{-1}(m)$. Then $\mu^{-1}(m)$ is a finite set with k points.*

Sometimes one has to deal with a local notion of degree. The main result is the following one [1, p. 38].

Theorem 2. *Let Ω be a bounded open set in \mathbb{R}^n . Let $\mu: \overline{\Omega} \rightarrow \mathbb{R}^n$ be a map which is continuous on the closure $\overline{\Omega}$ and differentiable on Ω . Suppose that 0 is*

a regular value of μ and that $0 \notin \mu(\partial\Omega)$. Then

$$\deg(\mu, \Omega, 0) = \sum_{\lambda \in \mu^{-1}(0)} \operatorname{sgn}[J_\mu(\lambda)]$$

where we denote by J_μ the Jacobian of μ .

A well known consequence is that if $\deg(\mu, \Omega, 0) \neq 0$ then the equation $\mu(\lambda) = 0$ has at least one solution in Ω . In fact, for maps of the sphere into itself, the *existence* of solutions only depends on the *continuity*, by the following result [20, Ch. VIII, Ex. 2.5, p. 191]:

Proposition 3. *Let $\mu: S^4 \rightarrow S^4$ be a continuous map whose degree is nonzero. Then μ is surjective.*

2.2 Linearization

We consider the space of quaternions \mathbb{H} as a differentiable manifold diffeomorphic to \mathbb{R}^4 . Then the differential $\mu_{*\lambda}: \mathbb{H} \rightarrow \mathbb{H}$ at the point $\lambda \in \mathbb{H}$ of the differentiable map $\mu: \mathbb{H} \rightarrow \mathbb{H}$ can be computed by means of the formula

$$\mu_{*\lambda}(X) = \frac{d}{dt}\bigg|_{t=0} \mu(\lambda + tX) = \lim_{t \rightarrow 0} \frac{1}{t} (\mu(\lambda + tX) - \mu(\lambda)).$$

Lemma 4. 1. *Let $f, g: \mathbb{H} \rightarrow \mathbb{H}$ be two differentiable maps. Then the differential of the product is given by*

$$(f \cdot g)_{*\lambda}(X) = f_{*\lambda}(X) \cdot g(\lambda) + f(\lambda) \cdot g_{*\lambda}(X);$$

2. *assume that $f(\lambda) \neq 0$ for all $\lambda \in \mathbb{H}$. Let $1/f: \mathbb{H} \rightarrow \mathbb{H}$ be the map given by $(1/f)(\lambda) = f(\lambda)^{-1}$. Then*

$$(1/f)_{*\lambda}(X) = -f(\lambda)^{-1} f_{*\lambda}(X) f(\lambda)^{-1}.$$

2.3 Sylvester equation

Let $P, Q, R \in \mathbb{H}$ be three quaternions. We are interested (see formula (6)) in the rank of \mathbb{R} -linear maps $\Sigma: \mathbb{H} \rightarrow \mathbb{H}$ of the form $\Sigma(X) = PX + XQ$. The equation $\Sigma(X) = R$ has been widely studied, sometimes under the name of Sylvester equation [17].

Lemma 5. 1. *Let $P = t + xi + yj + zk$. Then the matrix associated to the \mathbb{R} -linear map $X \mapsto PX$ with respect to the basis $\{1, i, j, k\}$ is $L(P) = \Re(P) \operatorname{Id} + A(P)$, where $\Re(P)$ is the real part of P and*

$$A(P) = \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & -z & y \\ y & z & 0 & -x \\ z & -y & x & 0 \end{bmatrix};$$

2. analogously, if $Q = s + u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ then the matrix associated to the right translation $X \mapsto XQ$ is $R(Q) = \Re(Q) \text{Id} + B(Q)$, where

$$B(Q) = \begin{bmatrix} 0 & -u & -v & -w \\ u & 0 & w & -v \\ v & -w & 0 & u \\ w & v & -u & 0 \end{bmatrix}.$$

Next Theorem is a reformulation of the results by Janovská and Opfer in [15], see also [11].

Theorem 6. 1. The rank of Σ is even, namely 0, 2 or 4;

2. $\text{rank } \Sigma < 4$ if and only if P and $-Q$ are similar quaternions, i.e. they have the same norm and the same real part;

3. $\text{rank } \Sigma = 0$ if and only if P is a real number and $Q = -P$.

Proof. The matrix associated to Σ is

$$J = \begin{bmatrix} t+s & -x-u & -y-v & -z-w \\ x+u & t+s & -z+w & y-v \\ y+v & z-w & t+s & -x+u \\ z+w & -y+v & x-u & t+s \end{bmatrix}.$$

Since

$$\det J = (t+s)^4 + 2(t+s)^2(x^2 + y^2 + z^2 + u^2 + v^2 + w^2) + (x^2 + y^2 + z^2 - u^2 - v^2 - w^2)^2 \geq 0, \quad (1)$$

the matrix J has rank 4 excepting when $t+s = 0$ and $x^2 + y^2 + z^2 = u^2 + v^2 + w^2$. In this case J is skew-symmetric, hence its rank is even. If $\text{rank } \Sigma = 0$ it follows that $x = y = z = 0$ and $u = v = w = 0$. \square

2.4 Resolution of arbitrary linear equations

More generally, let us consider a linear equation of the form

$$P_1 X Q_1 + \cdots + P_n X Q_n = R, \quad \text{with } P_i, Q_i, R \in \mathbb{H}. \quad (2)$$

For each bilateral term $X \mapsto PXQ$, the matrices $L(P)$ and $R(Q)$ commute, because $P(XQ) = (PX)Q$. Then $A(P)$ and $B(Q)$ commute too. This implies that the quaternionic linear equation (2) is equivalent to the real linear system $MX = R$, where $X, R \in \mathbb{R}^4$ and M is the 4×4 real matrix

$$M = \sum_i L(P_i)R(Q_i) = \sum_i a_i b_i \text{Id} + \sum_i (a_i B_i + b_i A_i) + \sum_i A_i B_i$$

with $a_i = \Re(P_i)$, $b_i = \Re(Q_i)$, $A_i = A(Q_i)$ and $B_i = B(Q_i)$. Contrary to the case $n = 2$, when $n \geq 3$ the rank of M may be odd.

Example 7. The rank of the matrix associated to the bilateral linear equation $\mathbf{k}X + X(2 - \mathbf{i}) - 2\mathbf{j}X\mathbf{j} = 0$ is 3.

2.5 Determinants

It is possible to generalize to the quaternions the norm $|\det|$ (that is, with real values) of the complex determinant. Papers [2, 3] are surveys of the general theory of quaternionic determinants. For the relationship between Study's determinant and quasideterminants see [10, pp. 76–85].

Definition 8. Let the quaternionic matrix $A \in \mathcal{M}_{n \times n}(\mathbb{H})$ be decomposed as $A = X + \mathbf{j}Y$ with $X, Y \in \mathcal{M}_{n \times n}(\mathbb{C})$. We shall call *Study's determinant* of A the non-negative real number

$$\text{Sdet}(A) := (\det c(A))^{1/2},$$

where $c(A)$ is the complex matrix $c(A) = \begin{bmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{bmatrix} \in \mathcal{M}_{2n \times 2n}(\mathbb{C})$.

Remark 9. Up to the exponent, this is the same determinant which appears in Theorem 8.1 of [26], as well as others considered in [2]. We have normalized the exponent to $1/2$ in order to ensure that $\text{Sdet}(D) = |q_1 \dots q_n|$ for diagonal matrices $D = \text{diag}(q_1, \dots, q_n)$.

Proposition 10 ([3]). *Sdet is the only functional that verifies the properties:*

1. $\text{Sdet}(AB) = \text{Sdet}(A) \cdot \text{Sdet}(B)$;
2. if A is a complex matrix then $\text{Sdet}(A) = |\det(A)|$.

The following immediate consequences are very useful for computations.

- Corollary 11.**
1. $\text{Sdet}(A) > 0$ if and only if the matrix A is invertible;
 2. let A and $B = PAP^{-1}$ be similar matrices, then $\text{Sdet}(A) = \text{Sdet}(B)$;
 3. $\text{Sdet}(A)$ does not change when a (right) multiple of one column is added to another column;
 4. $\text{Sdet}(A)$ does not change when a (left) multiple of one row is added to another row;
 5. $\text{Sdet}(A)$ does not change when two columns (or two rows) are permuted.

We shall need the following result too (we have not found it explicitly in the literature):

Proposition 12. For any matrix with two boxes A, B of order m and n respectively, it holds that $\text{Sdet} \begin{bmatrix} A & 0 \\ * & B \end{bmatrix} = \text{Sdet}(A) \cdot \text{Sdet}(B)$.

2.6 Jacobi identity

Let C be a complex $n \times n$ matrix. Let $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_p\}$ be two subsets of $\{1, \dots, n\}$ with the same size p . Let us denote by $C_{I,J}$ the submatrix formed by the rows with index in I and the columns with index in J . On the other side, let us denote by $C^{I,J}$ the complementary matrix obtained by suppressing the rows in I and the columns in J .

The following *Jacobi identity* is attributed to Kronecker in [22].

Lemma 13. *Assume that the complex matrix C is invertible. Then*

$$\det(C^{-1})_{I,J} = (-1)^{I+J} \det C^{J,I} / \det C,$$

where $I + J$ means $i_1 + \dots + i_p + j_1 + \dots + j_p$.

A generalization to quasideterminants appears in [10, Theorem 1.5.4, p. 74], see also Section 3.2. We shall use Study's determinant to establish an analogous result in the quaternionic setting.

Proposition 14. *Let A be an invertible quaternionic matrix. Then*

$$\text{Sdet}(A^{-1})_{I,J} = \text{Sdet } A^{J,I} / \text{Sdet } A.$$

Proof. If $I = \{i_1, \dots, i_p\}$ we denote $I' = I + n = \{i_1 + n, \dots, i_p + n\}$; analogously $J' = J + n$. The result follows from Lemma 13 because

$$c((A^{-1})_{I,J}) = c(A^{-1})_{I \cup I', J \cup J'} = (c(A)^{-1})_{I \cup I', J \cup J'}$$

and

$$c(A^{J,I}) = c(A)^{J \cup J', I \cup I'}.$$

□

3 CHARACTERISTIC EQUATION

The problem we are proposing here is to find a characteristic map for the left eigenvalues of a given matrix A , that is to find a map $\mu_A: \mathbb{H} \rightarrow \mathbb{H}$ such that $\mu_A(\lambda) = 0$ if and only if λ is a left eigenvalue of A . Notice that the function $\text{Sdet}(A - \lambda \text{Id})$ is real-valued, so it is not of interest from the point of view of the topological degree, nor it is solvable in any obvious way.

3.1 Left eigenvalues

Let A be a matrix with quaternionic coefficients.

Definition 15. The quaternion $\lambda \in \mathbb{H}$ is a *left eigenvalue* of A if the matrix $A - \lambda \text{Id}$ is not invertible, or equivalently $\text{Sdet}(A - \lambda \text{Id}) = 0$.

Let $\sigma_l(A)$ be the left spectrum, i.e. the set of left eigenvalues, of the matrix A .

Proposition 16. *The spectrum $\sigma_l(A)$ is compact.*

Proof. The spectrum is a closed set because it is given by the equation $\text{Sdet}(A - \lambda \text{Id}) = 0$. It is bounded because $\lambda \in \sigma_l(A)$ if and only if there exists $v \in \mathbb{H}^n$, $v \neq 0$, such that $Av = \lambda v$; then

$$|\lambda| = \frac{|\lambda v|}{|v|} \leq \sup_{|w|=1} \frac{|Aw|}{|w|} = |A|.$$

□

Proposition 17. *Let B be an invertible matrix. Then $\lambda \in \sigma_l(B)$ if and only if $\lambda^{-1} \in \sigma_l(B^{-1})$.*

Proof. If $Bv = \lambda v$ then $B^{-1}(\lambda v) = B^{-1}Bv = v = \lambda^{-1}(\lambda v)$. □

3.2 Background

When the matrix A is hermitian, all left eigenvalues are real numbers so they coincide with the right eigenvalues [7]. Moreover it is possible to define a true determinant for hermitian matrices [18, 26], which allows to construct a characteristic polynomial $p(t) = \det(A - t \text{Id})$ with real variable.

For the general case, $\text{Sdet}(A)$ equals, up to an exponent, the determinant of AA^* . On the other hand, Zhang [26] pointed out that if the quaternionic matrix is decomposed as $A = X + \mathbf{j}Y$, with $X, Y \in \mathcal{M}_{n \times n}(\mathbb{C})$, then its left eigenvalues $\lambda = x + \mathbf{j}y$, with $x, y \in \mathbb{C}$, are the roots of the function $\sigma: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ given by

$$\sigma(x, y) = \det \begin{bmatrix} X - x \text{Id} & -\overline{Y} + \overline{y} \text{Id} \\ Y - y \text{Id} & \overline{X} - \overline{x} \text{Id} \end{bmatrix}. \quad (3)$$

This is equivalent to the equation $\text{Sdet}(A - \lambda \text{Id}) = 0$.

Another approach is due to Gelfand *et al.* [10]. To each matrix $A \in \mathcal{M}_{n \times n}(\mathbb{H})$, they associated n^2 functions, that we shall call *quasicharacteristic functions*, defined by

$$f_{ij}(\lambda) = |\lambda \text{Id} - A|_{ij}, \quad 1 \leq i, j \leq n,$$

where $|\cdot|_{ij}$ is the (i, j) -quasideterminant. Let us denote by $A^{i,j}$ the submatrix of order $(n-1)$ obtained by suppressing the row i and the column j in the matrix $A \in \mathcal{M}_{n \times n}(\mathbb{H})$. Then quasideterminants are defined inductively by the formula

$$|A|_{ij} = a_{ij} - \sum a_{iq} (|A^{i,j}|_{pq})^{-1} a_{pj},$$

where the sum is taken over the indices $p, q \in \{1, \dots, n\}$ with $p \neq i, q \neq j$, such that the quasideterminant of lower order $|A^{i,j}|_{pq}$ is defined and it is non-null (see Proposition 1.5 of [9]).

When the matrix A is invertible, the entries of the inverse matrix A^{-1} are exactly $a^{ij} = |A|_{ji}^{-1}$. In the commutative case this gives the well known formula $a^{ij} = (-1)^{i+j} \det A^{j,i} / \det A$. For quaternionic matrices, the norm of the quasideterminant $|A|_{ij}$ of $A \in \mathcal{M}_{n \times n}(\mathbb{H})$ verifies

$$||A|_{ij}| \cdot \text{Sdet}(A^{i,j}) = \text{Sdet}(A). \quad (4)$$

This is a particular case of Jacobi identity for quaternions (Proposition 14).

Remark 18. From Equation (4) it follows that the roots of the quasicharacteristic functions are left eigenvalues. However none of those functions gives the complete spectrum, as shown in the next Example. Also notice that the definition of *noncommutative left eigenvalue* considered in [10, subsection 8.2, p. 128] does not correspond to the notion we are discussing here.

Example 19. Let $A = \begin{bmatrix} \mathbf{i} & 0 & 0 \\ \mathbf{k} & \mathbf{j} & 0 \\ -3\mathbf{i} & 2\mathbf{k} & \mathbf{k} \end{bmatrix}$. Then $\sigma_l(A) = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The quasicharacteristic functions are

$$\begin{aligned} f_{11}(\lambda) &= \lambda - \mathbf{i}, \\ f_{12}(\lambda) &= -(\lambda - \mathbf{i})\mathbf{k}(\lambda - \mathbf{j}), \\ f_{13}(\lambda) &= -(\lambda - \mathbf{i})(3\mathbf{i} - 2\mathbf{k}(\lambda - \mathbf{j})^{-1}\mathbf{k})^{-1}(\lambda - \mathbf{k}), \\ f_{22}(\lambda) &= \lambda - \mathbf{j}, \\ f_{23}(\lambda) &= -\frac{1}{2}(\lambda - \mathbf{j})\mathbf{k}(\lambda - \mathbf{k}), \\ f_{32}(\lambda) &= -2\mathbf{k} - (\lambda - \mathbf{k})\mathbf{k}(\lambda - \mathbf{j}), \\ f_{33}(\lambda) &= \lambda - \mathbf{k}, \end{aligned}$$

while $f_{21}(\lambda)$ and $f_{31}(\lambda)$ are not defined.

3.3 Characteristic map

We now introduce the notion of a characteristic map whose roots are the left eigenvalues, thus generalizing the usual characteristic polynomial in the real and complex settings. As we shall see this notion fits naturally with the equation of order 2 given by Wood [25], as well as with the procedure proposed by So [24] in order to compute the left eigenvalues of 3×3 matrices.

Definition 20. The map $\mu : \mathbb{H} \rightarrow \mathbb{H}$ is a *characteristic map* of the matrix $A \in \mathcal{M}_{n \times n}(\mathbb{H})$ if, up to a constant, its norm verifies $|\mu(\lambda)| = \text{Sdet}(A - \lambda \text{Id})$ for all $\lambda \in \mathbb{H}$.

Example 21. Let $D = \text{diag}(q_1, \dots, q_n)$ be a diagonal matrix. Then $\mu(\lambda) = (q_1 - \lambda) \cdots (q_n - \lambda)$ is a characteristic map for D . Analogously for a triangular matrix.

Let us start with the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If A is a diagonal matrix, then $\sigma_l(A)$ reduces to the elements in the diagonal. Otherwise, we can always suppose that $b \neq 0$ (see Remark 24). Moreover, $\text{Sdet}(A - \lambda \text{Id})$ does not change after elementary transformations (Corollary 11), for instance

$$\text{Sdet} \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = \text{Sdet} \begin{bmatrix} 0 & b \\ c - (d - \lambda)b^{-1}(a - \lambda) & d - \lambda \end{bmatrix}.$$

Then, as pointed out by Wood, computing the left spectrum is equivalent to finding the roots of a characteristic map like

$$\mu(\lambda) = c - (d - \lambda)b^{-1}(a - \lambda). \quad (5)$$

Remark 22. Huang [12] proposed another map when $c \neq 0$, namely $(\lambda - a)c^{-1}(\lambda - d) - b$. This polynomial is obtained by adding $(\lambda - a)c^{-1}$ by the second row to the first row. This expression is equivalent to $b - (a - \lambda)c^{-1}(d - \lambda)$, which is the one given by Wood at the end of [25] (there is a misprint in the original paper).

The left spectrum is not invariant by similarity. However, we shall use the following fact:

Proposition 23. *If P is an invertible real matrix then $\text{Sdet}(A - \lambda \text{Id}) = \text{Sdet}(PAP^{-1} - \lambda \text{Id})$. Hence A and PAP^{-1} have the same characteristic maps.*

Remark 24. Let A be a matrix of order $n \geq 2$, let $P_{\alpha\beta}$ be the real matrix obtained by interchanging the rows α and β in the identity matrix I_n . Left (resp. right) multiplication by the matrix $P_{\alpha\beta}$ switches two rows (resp. columns) of A . Now let $i \neq j$ be two indices and let π be any permutation of $\{1, \dots, n\}$ sending i to 1 and j to n . Then π can be written as a composition of transpositions, so by taking the product P of the corresponding matrices $P_{\alpha\beta}$ we obtain the matrix PAP^{-1} where the initial entry a_{ij} of A has moved to the place $(1, n)$.

4 SPECTRUM OF MATRICES OF ORDER 2

In the next paragraphs we shall classify the different possible spectra of 2×2 quaternionic matrices depending on the rank of the differential $\mu_{*\lambda}$ of a characteristic map.

4.1 Preliminaries

The characteristic map $\mu: \mathbb{H} \rightarrow \mathbb{H}$ given in (5) can be extended to a continuous (or even differentiable) map $\mu: S^4 \rightarrow S^4$ on the sphere $S^4 = \mathbb{H} \cup \{\infty\}$, because $\lim |\mu(\lambda)| = \infty$ when $|\lambda| \rightarrow \infty$. A rigorous proof of the following result can be found in Eilenberg-Steenrod's book [6, pp. 304–310]:

Proposition 25. *A polynomial map like μ and the power map λ^2 are homotopic, hence they have the same topological degree, which equals 2.*

From Lemma 4 it follows that the differential of μ is given by

$$\mu_{*\lambda}(X) = Xb^{-1}(a - \lambda) + (d - \lambda)b^{-1}X. \quad (6)$$

4.2 Classification of left spectra

Now we are in a position to reformulate the following result from Huang and So [13], see also [21, 8]. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a quaternionic matrix with $b \neq 0$, and denote

$$a_0 = -b^{-1}c, \quad a_1 = b^{-1}(a - d), \quad \Delta = a_1^2 - 4a_0.$$

Theorem 26 ([13]). *The matrix A has one, two or infinite left eigenvalues. The latter case is equivalent to the following conditions: a_0, a_1 are real numbers such that $a_0 \neq 0$ and $\Delta < 0$.*

Remark 27. We shall call *spherical* the infinite case, because the spectrum $\sigma_l(A) = \{(1/2)(a + d + bq) : q^2 = \Delta\}$ is diffeomorphic to the sphere $S^2 \subset \mathbb{H}_0 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$.

Let $\lambda \in \mathbb{H}$ be an eigenvalue of A , that is $\mu(\lambda) = 0$ for the map μ in (5). In the next Propositions we shall apply Theorem 6 to the differential $\Sigma = \mu_{*\lambda}$ computed in (6). Accordingly to the notation of Sylvester equation in Section 2.3, we denote

$$P = (d - \lambda)b^{-1}, \quad Q = b^{-1}(a - \lambda).$$

First we study the two non-generic cases.

Proposition 28. *If $\text{rank } \mu_{*\lambda} = 0$, then a_0, a_1 are real numbers and $\Delta = 0$. Moreover λ equals $(a + d)/2$ and this is the only left eigenvalue of the matrix.*

Proof. We know from Theorem 6 that $P = t \in \mathbb{R}$ and $Q = -t$, then $a_1 = -2t$ and $2\lambda = a + d$. From $\mu(\lambda) = 0$ it follows that $a_0 = +t^2$, then $\Delta = 0$. Now it is easy to check (using for instance Theorem 2.3 in [13]) that $\lambda = a + tb$ is the only left eigenvalue of A . \square

Lemma 29. *Let A, B be two similar quaternions that do not commute. Then the equation $\lambda^2 - (A + B)\lambda + AB = 0$ has the unique solution $\lambda = B$.*

Proof. If $\lambda \neq B$ is a solution, it follows from $(\lambda - B)\lambda = A(\lambda - B)$ that λ and A are similar, then $\Re(\lambda) = \Re(A) = \Re(B)$ and $|\lambda| = |A| = |B|$. By substituting in the equation we see that the real parts and norms disappear, so we can suppose that A, B, λ are pure imaginary quaternions with norm 1. Hence $\lambda^2 = -1 = B^2$ so the equation reduces to $(A + B)\lambda = (A + B)B$, which implies $\lambda = B$, a contradiction.

Alternatively, the uniqueness of λ can be proved by using Theorem 2.3, case 4.1, of the solution of quadratic equations in [14]. \square

Proposition 30. *If $\text{rank } \mu_{*\lambda} = 2$ two things may happen:*

1. either the spectrum is spherical and all the eigenvalues have rank 2;
2. or the matrix has just one eigenvalue.

Proof. By using the diffeomorphism $a + b\sigma_l(A') = \sigma_l(A)$ we can substitute A by the so-called “companion matrix” $A' = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$. Since the rank is 2, we have from Theorem 6 that $P = t + \alpha$ and $Q = -t + \beta$ with $\alpha, \beta \in \mathbb{H}_0 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$, $|\alpha| = |\beta| \neq 0$. Then $a_1 = -2t + \beta - \alpha$. The first possibility is that $\beta = \alpha$, then $a_1 = -2t$. It follows from $\mu(\lambda) = 0$ that $a_0 = t^2 + |\beta|^2 \neq 0$ and $\Delta = -4|\beta|^2 < 0$. Then we have the spherical case. In particular $\lambda = (-a_1 - 2\beta)/2$. The other eigenvalues have the form $(-a_1 + q)/2$ with $q^2 = -4|\beta|^2$, then the differential of μ verifies $P = t - q^{-1}/2$ and $Q = -t - q/2$, and so they have rank 2 too. The second possibility is that $\beta \neq \alpha$. Then $a_1 = -2t + \beta - \alpha$, $a_0 = (t + \alpha)(t - \beta)$, and Lemma 29 shows that the only eigenvalue is $\lambda = t - \beta$. \square

Now we consider the generic case.

Proposition 31. *If $\text{rank } \mu_{*\lambda} = 4$ then the matrix has two different eigenvalues.*

Proof. Since the differential has maximal rank at the eigenvalue λ , the matrix A cannot correspond to Propositions 28 or 30, hence all its eigenvalues are of rank 4. Then by the inverse function theorem the fiber $\mu^{-1}(0)$ is discrete (in fact compact) and its cardinal equals (Theorem 1) the degree of the map μ , which is 2 by Proposition 25. Notice that the Jacobian is nonnegative by formula (1). \square

Remark 32. In [16], Janovská and Opfer show that for quaternionic polynomials there are several types of zeros accordingly to the rank of some real 4×4 matrix, but their procedure does not seem to have an immediate geometrical meaning.

5 CHARACTERISTIC MAPS OF 3×3 MATRICES

The only known results about the left spectrum of 3×3 matrices are due to So [24], who did a case by case study, depending on some relationships among the entries of the matrix. He showed that when $n = 3$ left eigenvalues could be found by solving quaternionic polynomials of degree not greater than 3. In general there is not any known method for solving the resulting equations.

In the following paragraphs we shall develop a method for matrices of order 3 which is analogous to that of Section 4, that is we shall find a map μ_A such that $|\mu_A(\lambda)| = \text{Sdet}(A - \lambda \text{Id})$. This time, however, the characteristic map μ_A will be in most cases a rational function instead of a polynomial (the latter occurs when the matrix A has some null entry outside the diagonal).

Let us consider the quaternionic matrix $A = \begin{bmatrix} a & b & c \\ f & g & h \\ p & q & r \end{bmatrix} \in \mathcal{M}_{3 \times 3}(\mathbb{H})$.

5.1 Polynomial case

We start studying the simplest situation, when there exists some zero entry outside the diagonal.

First, suppose that the matrix has the zero entry $c = 0$, that is

$$\text{Sdet}(A - \lambda \text{Id}) = \text{Sdet} \begin{bmatrix} a - \lambda & b & 0 \\ f & g - \lambda & h \\ p & q & r - \lambda \end{bmatrix}.$$

There are three possibilities:

1. if $b, h = 0$, we have a triangular matrix, so we can take

$$\mu(\lambda) = (r - \lambda)(g - \lambda)(a - \lambda); \quad (7)$$

2. if $b = 0$ but $h \neq 0$, then Proposition 12 allows us to reduce to the 2×2 case and we obtain

$$\mu(\lambda) = (q - (r - \lambda)h^{-1}(g - \lambda))(a - \lambda); \quad (8)$$

3. finally, if $b \neq 0$, we can proceed as follows. We create a zero in the first row by subtracting to the first row C_1 the multiple $C_2 b^{-1}(a - \lambda)$ of the second column:

$$\text{Sdet}(A - \lambda \text{Id}) = \text{Sdet} \begin{bmatrix} 0 & b & 0 \\ f - (g - \lambda)b^{-1}(a - \lambda) & g - \lambda & h \\ p - qb^{-1}(a - \lambda) & q & r - \lambda \end{bmatrix},$$

then we permute the two last columns in order to reduce to the 2×2 case.

In this way we can take as a characteristic map the polynomial of degree 3

$$\mu(\lambda) = p - qb^{-1}(a - \lambda) - (r - \lambda)h^{-1}(f - (g - \lambda)b^{-1}(a - \lambda)). \quad (9)$$

Theorem 33. *If the matrix $A \in \mathcal{M}_{3 \times 3}(\mathbb{H})$ has some zero entry outside the diagonal, then A admits a polynomial characteristic map.*

Proof. Let the entry be $a_{ij} = 0$, with $i \neq j$. Then accordingly to Remark 24 there is a real invertible matrix P such that the transformation PAP^{-1} does not change the characteristic maps and gives a matrix with $a_{13} = 0$. \square

5.2 Rational case

In the more general situation, when $c \neq 0$, we can compute the Study's determinant of the matrix A by creating zeroes in the first row. Then

$$\text{Sdet}(A) = \text{Sdet} \begin{bmatrix} 0 & 0 & c \\ f - hc^{-1}a & g - hc^{-1}b & h \\ p - rc^{-1}a & q - rc^{-1}b & r \end{bmatrix}.$$

From Lemma 12 and the results for 2×2 matrices it follows:

Proposition 34. *If $c \neq 0$, then $\text{Sdet}(A)$ is given:*

1. *when $g - hc^{-1}b \neq 0$, by*

$$|c| \cdot |g - hc^{-1}b| \cdot |p - rc^{-1}a - (q - rc^{-1}b)(g - hc^{-1}b)^{-1}(f - hc^{-1}a)|;$$

2. *otherwise, by*

$$|c| \cdot |q - rc^{-1}b| \cdot |f - hc^{-1}a|.$$

Definition 35. We shall call *pole* of the matrix $A \in \mathcal{M}_{3 \times 3}(\mathbb{H})$ the point $\pi_A = g - hc^{-1}b$.

Notice that π_A is the quasideterminant $|A^{3,1}|_{21}$ (see page 7).

By applying Proposition 34 to the matrix $A - \lambda \text{Id}$ we obtain the following characteristic map for A (we omit the term $|c|$).

Proposition 36. *Let A be a matrix of order 3×3 such that $c \neq 0$. A characteristic map can be defined as follows:*

1. *if $\pi_A = g - hc^{-1}b$ is the pole of A , then*

$$\mu(\pi_A) = (q - (r - \pi_A)c^{-1}b) (f - hc^{-1}(a - \pi_A));$$

2. *for $\lambda \neq \pi_A$ we define*

$$\begin{aligned} \mu(\lambda) = & (\pi_A - \lambda) [p - (r - \lambda)c^{-1}(a - \lambda) - \\ & (q - (r - \lambda)c^{-1}b) (\pi_A - \lambda)^{-1} (f - hc^{-1}(a - \lambda))] . \end{aligned} \quad (10)$$

Remark 37. The map in (10) is exactly the same formula given by So in [24, p. 563], even if our method is completely different. This is why we chose to compute determinants starting from the right bottom corner.

5.3 Continuity at the pole

Up to now we have defined maps which verify $|\mu(\lambda)| = \text{Sdet}(A - \lambda \text{Id})$ in norm. Since Sdet is a continuous map we have $\lim_{\lambda \rightarrow \pi_A} |\mu(\lambda)| = |\mu(\pi_A)|$. However, the following example shows that μ may not be continuous at the pole π_A .

Example 38. Let $A = \begin{bmatrix} 0 & \mathbf{i} & 1 \\ 3\mathbf{i} - \mathbf{k} & 0 & 1 \\ \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & 0 \end{bmatrix}$. The pole $\pi_A = -\mathbf{i}$ is not a left eigenvalue; in fact

$$\mu(\pi_A) = (-1 + \mathbf{j} + \mathbf{k} + 1)(3\mathbf{i} - \mathbf{k} - \mathbf{i}) = 1 - \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}.$$

We observe that the directional limits

$$\lim_{\varepsilon \rightarrow 0} \mu(-\mathbf{i} + \varepsilon q) = -q(\mathbf{j} + \mathbf{k})q^{-1}(2\mathbf{i} - \mathbf{k})$$

depend on $q \in \mathbb{H}$, hence $\lim_{\lambda \rightarrow \pi_A} \mu(\lambda)$ does not exist.

Theorem 39. *The characteristic rational map μ_A is continuous if and only if the pole π_A is a left eigenvalue of A .*

Proof. Assume that π_A is a left eigenvalue. Let $(q_n)_n$ be a sequence converging to π_A . Then $|\mu(q_n)| = \text{Sdet}(A - q_n \text{Id})$ converges to $\text{Sdet}(A - \pi_A \text{Id}) = 0$, that is $\mu(q_n) \rightarrow 0 = \mu(\pi_A)$.

Now we shall prove the converse. The map μ defined in Proposition 36 is of the form

$$\mu(\lambda) = (\pi_A - \lambda) [p(\lambda) - q(\lambda)(\pi_A - \lambda)^{-1}f(\lambda)], \quad \lambda \neq \pi_A, \quad (11)$$

while $\mu(\pi_A) = q(\pi_A)f(\pi_A)$. Let us assume that $\lim_{\lambda \rightarrow \pi_A} \mu(\lambda)$ exists and equals $\mu(\pi_A)$. We must check that $\mu(\pi_A) = 0$. If $f(\pi_A) = 0$ we have finished. Otherwise we deduce from (11) that

$$\lim_{\lambda \rightarrow \pi_A} (\lambda - \pi_A)q(\lambda)(\lambda - \pi_A)^{-1} = -q(\pi_A). \quad (12)$$

From Lemma 40 it follows that the limit on the left side equals $q(\pi_A)$, hence $q(\pi_A) = 0$, which ends the proof. \square

Lemma 40. *Let $Q = Q(\lambda)$ be a continuous map and suppose that there exists the limit $l_0 = \lim_{\lambda \rightarrow 0} \lambda Q(\lambda)\lambda^{-1}$. Then $l_0 = Q(0)$.*

Proof. Take a sequence of real numbers $(\varepsilon_n)_n \rightarrow 0$. Then

$$l_0 = \lim \varepsilon_n Q(\varepsilon_n) \varepsilon_n^{-1} = \lim Q(\varepsilon_n) = Q(0).$$

\square

Notice that the differentiability of μ at the pole π_A is not ensured.

It is an open question whether it is always possible or not to find a polynomial, or at least a continuous characteristic map for a given matrix A .

5.4 Extension to the infinite point

Each of the characteristic maps we have introduced up to now can be extended to the sphere $S^4 = \mathbb{H} \cup \{\infty\}$.

Proposition 41. *The polynomial maps μ defined in Subsection 5.1 verify that $\lim_{|\lambda| \rightarrow \infty} |\mu(\lambda)| = \infty$.*

Proof. When $b = 0$, the result follows from formulae (7) and (8); when $b \neq 0$, the expression of μ is that of formula (9), so

$$\frac{|\mu(\lambda)|}{|\lambda|^2} \geq \frac{|(r - \lambda)h^{-1}(g - \lambda)b^{-1}(a - \lambda)|}{|\lambda|^2} - \frac{|-p + qb^{-1}(a - \lambda) + (r - \lambda)h^{-1}f|}{|\lambda|^2}.$$

\square

Proposition 42. *The rational map μ defined in the $c \neq 0$ case by formula (10) can be extended to $S^4 = \mathbb{H} \cup \{\infty\}$ (maybe with a discontinuity at the pole π_A).*

Proof. We have

$$\mu(\lambda) = (\pi_A - \lambda)p_2(\lambda) - (\pi_A - \lambda)q_1(\lambda)(\pi_A - \lambda)^{-1}f_1(\lambda)$$

with polynomials

$$\begin{aligned} p_2(\lambda) &= p - (r - \lambda)c^{-1}(a - \lambda), \\ q_1(\lambda) &= q - (r - \lambda)c^{-1}b, \\ f_1(\lambda) &= f - hc^{-1}(a - \lambda). \end{aligned} \tag{13}$$

Then

$$|\mu(\lambda)| \geq |(\pi_A - \lambda)p_2(\lambda)| - |q_1(\lambda)f_1(\lambda)|$$

and

$$\lim_{|\lambda| \rightarrow \infty} \frac{|\mu(\lambda)|}{|\lambda|^3} \geq \lim_{|\lambda| \rightarrow \infty} \frac{|(\pi_A - \lambda)p_2(\lambda)|}{|\lambda|^3} = \lim_{|\lambda| \rightarrow \infty} |c^{-1}| \frac{|r - \lambda|}{|\lambda|} \frac{|a - \lambda|}{|\lambda|} = |c^{-1}|.$$

□

5.5 Discontinuous case

Let us now assume that the characteristic map μ_A defined in Section 36 is not continuous at the pole π_A , or equivalently that π_A is not a left eigenvalue of A (Theorem 39). Then the matrix $B = A - \pi_A \text{Id}$ is invertible and its pole is $\pi_B = 0$. Moreover $\sigma_l(A) = \sigma_l(B) + \pi_A$. On the other hand, from Proposition 17 we know that the spectra of B and B^{-1} are diffeomorphic, because $\sigma_l(B^{-1}) = \sigma_l(B)^{-1}$.

Theorem 43. *Let A be a matrix such that the pole π_A is not a left eigenvalue. Let $B = A - \pi_A \text{Id}$. Then the matrix B^{-1} has a polynomial characteristic map.*

Proof. Accordingly to formula (4) the norm of the entry (1, 3) of the matrix B^{-1} equals

$$\text{Sdet}(B^{3,1}) / \text{Sdet}(B) = |\pi_B| / \text{Sdet}(B) = 0,$$

then Theorem 33 applies. □

Here is an alternative proof of Theorem 43. Let $\mu_B(\lambda) = -\lambda R(\lambda)$, with

$$R(\lambda) = p(\lambda) + q(\lambda)\lambda^{-1}f(\lambda), \quad \lambda \neq 0,$$

be the characteristic map given in (36) (we assume $\pi_B = 0$.) Then it is immediate that $\lambda R(\lambda^{-1})\lambda$ is a polynomial in λ (of degree 3 and independent term $-c^{-1}$) and we only have to apply the following result.

Proposition 44. *Let μ_B be a characteristic map of the invertible matrix $B = A - \pi_A \text{Id}$, with a discontinuity at the pole $\pi_B = 0$. Then*

$$\mu_{B^{-1}}(\lambda) = \text{Sdet}(B)^{-1} \lambda^2 \mu_B(\lambda^{-1}) \lambda, \quad \lambda \neq 0,$$

is a characteristic map for B^{-1} .

Proof. From Proposition 10 we have

$$\text{Sdet}(\lambda^{-1} \text{Id}) \cdot \text{Sdet}(B^{-1} - \lambda \text{Id}) \cdot \text{Sdet}(B) = \text{Sdet}(\lambda^{-1} \text{Id} - B)$$

then

$$|\lambda^{-3}| \cdot \text{Sdet}(B^{-1} - \lambda \text{Id}) \cdot \text{Sdet}(B) = |\mu_B(\lambda^{-1})|.$$

□

Remark 45. The idea that a rational map like $R(\lambda)$ can be converted into a polynomial $\lambda^{-1} R(\lambda) \lambda^{-1}$ with variable λ^{-1} is due to So (see [24, Lemma 3.5, p. 558]).

6 TOPOLOGICAL STUDY OF THE 3×3 CASE

We shall consider separately the polynomial, rational continuous and discontinuous cases considered in Section 5.

6.1 Polynomial case

Let us start with matrices having some null entry outside the diagonal (as we have seen this case can be reduced to the case $c = 0$). We know that the characteristic map is a polynomial of degree 3 which can be extended in a continuous way to the sphere S^4 . Then since there is a unique term of higher degree 3, the map μ_A is homotopic to λ^3 , so it has topological degree 3.

Proposition 46. *Let λ be a left eigenvalue of the matrix A with $c = 0$. Then the differential of the polynomial characteristic map μ_A in Subsection 5.1 is given by*

1. *if $b, h = 0$ then*

$$\mu_{*\lambda}(X) = -X(g - \lambda)(a - \lambda) - (r - \lambda)(g - \lambda)X - (r - \lambda)X(a - \lambda);$$

2. *if $b = 0, h \neq 0$ then*

$$\begin{aligned} \mu_{*\lambda}(X) = \\ Xh^{-1}(g - \lambda)(a - \lambda) - (q - (r - \lambda)h^{-1}(g - \lambda))X + (r - \lambda)h^{-1}X(a - \lambda); \end{aligned}$$

3. *otherwise,*

$$\begin{aligned} \mu_{*\lambda}(X) = & (qb^{-1} - (r - \lambda)h^{-1}(g - \lambda)b^{-1})X \\ & + Xh^{-1}(f - (g - \lambda)b^{-1}(a - \lambda)) - (r - \lambda)h^{-1}Xb^{-1}(a - \lambda). \end{aligned}$$

The proof is a direct application of the derivation rules given in Lemma 4.

The expressions obtained are of the form $PX + XQ + RXS = 0$, whose rank can be computed with the method given in Section 2.4.

Example 47. Let $A = \begin{bmatrix} a & 0 & 0 \\ f & g & 0 \\ p & q & r \end{bmatrix}$ be a triangular matrix. The differential $\mu_{*\lambda}(X)$ of the characteristic map

$$\mu(\lambda) = (r - \lambda)(g - \lambda)(a - \lambda),$$

at the eigenvalues $\lambda = a, g, r$ is given, respectively, by $(a - r)(g - a)X$, $(g - r)X(a - g)$ and $(r - g)(a - r)X$. Hence, unlike the case $n = 2$, the rank depends on the multiplicity of each eigenvalue, and can be either 0 or 4.

6.2 Rational case

When none of the entries outside the diagonal is zero, the characteristic map is a rational function, with a distinguished point π_A .

Let us first suppose that the pole π_A is a left eigenvalue. We know from (10) that μ_A is a continuous map on S^4 of the form

$$(\pi_A - \lambda) [p(\lambda) - q(\lambda)(\pi_A - \lambda)^{-1}f(\lambda)].$$

By examining formulae (13) it is clear that $p(\lambda)$ is homotopic to $-\lambda^2$ by the homotopy

$$tp - (tr - \lambda)(1 - t + tc^{-1})(ta - \lambda), \quad t \in [0, 1].$$

Analogously $q(\lambda) \sim \lambda$ and $f(\lambda) \sim \lambda$, so $\mu_A(\lambda)$ is homotopic to

$$(\pi_A - \lambda) [-\lambda^2 - \lambda(\pi_A - \lambda)^{-1}\lambda]$$

(notice that this map is continuous at $\lambda = 0$), which in turn is homotopic to $\lambda^3 - \lambda^2$ by the homotopy

$$(t\pi_A - \lambda) [-\lambda^2 - \lambda(t\pi_A - \lambda)^{-1}\lambda].$$

Finally the homotopy $\lambda^2(\lambda - t)$ shows that the map μ_A is homotopic to λ^3 . All these homotopies can be extended to the infinity.

Hence we have proved

Proposition 48. *When the rational characteristic map μ_A is continuous it has topological degree 3.*

On the other hand, suppose that π_A is not a left eigenvalue. Then the polynomial case applies to $(A - \pi_A \text{Id})^{-1}$ by Theorem 43. So we do not have to use the local theory of degree, whose main difficulty is the need of considering homotopies which are *admissible* with respect to the domain Ω of definition, see [1, p. 28].

Corollary 49. *Any 3×3 quaternionic matrix has at least one left eigenvalue.*

Proof. In all cases the eigenvalues (or its inverses) can be computed as the roots of a continuous map μ of degree 3; then, $\mu^{-1}(0)$ is not void (see Section 2). \square

6.3 Final remarks

In order to simplify the computation of the rank, by taking $B = A - \pi_A \text{Id}$ we can always assume that the pole is $\pi_B = 0$.

Proposition 50. *For a 3×3 matrix with $c \neq 0$ the differential of the characteristic map μ given in formula (10) at the point $\lambda \neq \pi_B = 0$ is*

$$\begin{aligned} \mu_{*\lambda}(X) = & X \left[-p + (r - \lambda)c^{-1}(a - \lambda) + (q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}(f - hc^{-1}(a - \lambda)) \right] \\ & + (-\lambda)Xc^{-1}(a - \lambda) - (-\lambda)Xc^{-1}b(-\lambda)^{-1}(f - hc^{-1}(a - \lambda)) \\ & - (-\lambda)(q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}X(-\lambda)^{-1}(f - hc^{-1}(a - \lambda)) \\ & + [(-\lambda)(r - \lambda)c^{-1} - (-\lambda)(q - (r - \lambda)c^{-1}b)(-\lambda)^{-1}hc^{-1}] X. \end{aligned}$$

When μ_A is continuous but not differentiable at the pole π_A the rank at π_A could be computed by taking a different characteristic map $\mu_{PAP^{-1}}$. In particular, by moving around the entries outside the diagonal of a given matrix (see Remark 24 of Section 5.2) one can obtain up to six different characteristic maps, each one with a different pole.

It is an open question whether there exists a matrix A verifying that all the poles of the matrices PAP^{-1} , with P real, are eigenvalues. If such an example does not exist then the non-polynomial case would not be necessary.

7 EXAMPLES

Here are some miscellaneous examples.

Example 51. Discontinuous map. Let A be the matrix given in Example 38. Then

$$B = A - \pi_A \text{Id} = \begin{bmatrix} \mathbf{i} & \mathbf{i} & 1 \\ 3\mathbf{i} - \mathbf{k} & \mathbf{i} & 1 \\ \mathbf{k} & -1 + \mathbf{j} + \mathbf{k} & \mathbf{i} \end{bmatrix}$$

is an invertible matrix with pole $\pi_B = 0$. By computing the quasideterminants we obtain the inverse matrix

$$B^{-1} = \frac{1}{10} \begin{bmatrix} 4\mathbf{i} - 2\mathbf{k} & -4\mathbf{i} + 2\mathbf{k} & 0 \\ -1 - 3\mathbf{i} + 8\mathbf{j} - 6\mathbf{k} & 1 + 3\mathbf{i} - 3\mathbf{j} + \mathbf{k} & -5\mathbf{j} - 5\mathbf{k} \\ 11 + \mathbf{i} - 8\mathbf{j} - 8\mathbf{k} & -1 - \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} & -5\mathbf{j} + 5\mathbf{k} \end{bmatrix}.$$

Its polynomial characteristic map is

$$\begin{aligned} \mu_{B^{-1}}(\lambda) = & 10 - \lambda\mathbf{i} - 2\mathbf{i}\lambda - \frac{1}{10}\mathbf{i}\lambda(2\mathbf{i} - \mathbf{k})\lambda - \frac{1}{10}\lambda(1 + \mathbf{j} + 2\mathbf{k})\lambda - \frac{1}{100}\lambda(\mathbf{j} + \mathbf{k})\lambda(2\mathbf{i} - \mathbf{k})\lambda. \end{aligned} \tag{14}$$

On the other hand, the rational characteristic map for B is

$$\mu_B(\lambda) = -\lambda R(\lambda) = -\lambda [1 + \mathbf{k} + \mathbf{i}\lambda + \lambda\mathbf{i} - \lambda^2 + (\mathbf{j} + \mathbf{k} + \lambda\mathbf{i})\lambda^{-1}(2\mathbf{i} - \mathbf{k} + \lambda)]$$

and one can check that $\lambda R(\lambda^{-1})\lambda$ equals (up to a constant) the map (14).

Example 52. An eigenvalue of rank 3. Let

$$A = \begin{bmatrix} \mathbf{j} & 1 & 0 \\ 2\mathbf{i} & -\mathbf{k} & 1 \\ 2 - \mathbf{i} - 2\mathbf{j} & -1 - \mathbf{j} + \mathbf{k} & -\mathbf{i} - \mathbf{k} \end{bmatrix}.$$

The characteristic map is

$$\mu(\lambda) = 2 - \mathbf{i} - 2\mathbf{j} + (1 + \mathbf{j} - \mathbf{k})(\mathbf{j} - \lambda) + (\mathbf{i} + \mathbf{k} + \lambda)(2\mathbf{i} + (\mathbf{k} + \lambda)(\mathbf{j} - \lambda)).$$

For the left eigenvalue $\lambda = 0$ the differential is

$$\mu_{*0}(X) = \mathbf{k}X + X\mathbf{i} + (\mathbf{i} + \mathbf{k})X\mathbf{j},$$

whose real associated matrix $M = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 \end{bmatrix}$ has rank 3.

Example 53. A matrix which can not be reduced to the polynomial case $c = 0$. We shall exhibit a matrix $A \in \mathcal{M}_{3 \times 3}(\mathbb{H})$ such that for any real invertible matrix P all entries in the matrix PAP^{-1} outside the diagonal are not null. Let $A = T + \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z$, with $T, X, Y, Z \in \mathcal{M}_{3 \times 3}(\mathbb{R})$. Then

$$PAP^{-1} = PTP^{-1} + \mathbf{i}PXP^{-1} + \mathbf{j}PYP^{-1} + \mathbf{k}PZP^{-1},$$

which means that the matrix PAP^{-1} has a null entry if and only if the same happens for the real matrices PTP^{-1} , PXP^{-1} , PYP^{-1} and PZP^{-1} . Moreover, from Remark 24 we can suppose that the null entry is at the place $(1, 3)$. Now, recall that A is the matrix associated to a linear map $\mathbb{H}^3 \rightarrow \mathbb{H}^3$ with respect to the canonical basis e_1, e_2, e_3 and that the real matrix $P = (p_{ij})$ represents the change to another basis v_1, v_2, v_3 , that is $v_j = \sum_i p^{ij} e_i$ where $P^{-1} = (p^{ij})$. Assume that T, X are the matrices associated to two rotations $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ with the same rotation axis \mathcal{L} and rotation angles $+\pi/2$ and $-\pi/2$ respectively. Then the nullity of the entry $(1, 3)$ means that $Tv_3, Xv_3 \in \langle v_2, v_3 \rangle$, which implies that either v_3 is in the direction of the axis, i.e. $\mathcal{L} = \langle v_3 \rangle$, or it is orthogonal to the axis, in which case $\langle v_2, v_3 \rangle = \mathcal{L}^\perp$ (otherwise it is impossible that v_3, Tv_3 and Xv_3 lie in the same plane). Now, suppose that Y, Z are two other rotations with axis \mathcal{L}' and rotation angles $\pm\pi/2$. If \mathcal{L} and \mathcal{L}' are different and not perpendicular, then it is impossible that $Yv_3, Zv_3 \in \langle v_2, v_3 \rangle$.

Example 54. A continuous rational characteristic map. The pole $\pi_A = 1 + \mathbf{j}$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 0 & -\mathbf{j} & \mathbf{i} \\ -1 + \mathbf{j} & \mathbf{j} & \mathbf{k} \\ p & q & r \end{bmatrix},$$

with $p, q, r \in \mathbb{H}$ arbitrary, $p, q \neq 0$. In fact,

$$\mu(\pi_A) = (q - (r - 1 - \mathbf{j})\mathbf{k})(-1 + \mathbf{j} + \mathbf{j}(-1 - \mathbf{j})) = 0.$$

Example 55. Generic polynomial case. Let

$$A = \begin{bmatrix} \mathbf{k} & 0 & 0 \\ 3\mathbf{i} - \mathbf{j} & -\mathbf{i} & \mathbf{i} \\ 1 - 2\mathbf{k} & \mathbf{j} & -\mathbf{j} \end{bmatrix}.$$

The characteristic map is $\mu(\lambda) = (-1 - \mathbf{k} + \lambda\mathbf{i})\lambda(\mathbf{k} - \lambda)$ hence $\sigma_l(A) = \{\mathbf{k}, 0, -\mathbf{i} - \mathbf{j}\}$. The differential of μ at each eigenvalue is

$$\begin{aligned} \mu_{*\mathbf{k}}(X) &= (-1 - \mathbf{i} + \mathbf{k})X, \\ \mu_{*0}(X) &= -(1 + \mathbf{k})X\mathbf{k}, \\ \mu_{*(-\mathbf{i}-\mathbf{j})}(X) &= X(1 + 2\mathbf{i} + \mathbf{k}). \end{aligned}$$

Then the matrix A has three different eigenvalues, all of them with maximal rank.

Example 56. Two eigenvalues, one of null rank, the other one of maximal rank. Let

$$A = \begin{bmatrix} -\mathbf{i} - \mathbf{j} & 0 & 0 \\ \mathbf{k} & -\mathbf{i} & \mathbf{i} \\ 1 - \mathbf{i} & \mathbf{j} & -\mathbf{j} \end{bmatrix}.$$

This time $\mu(\lambda) = (1 + \mathbf{k} - \lambda\mathbf{i})\lambda(\mathbf{i} + \mathbf{j} + \lambda)$ so $\sigma_l(A) = \{0, -\mathbf{i} - \mathbf{j}\}$. We have

$$\begin{aligned} \mu_{*0}(X) &= (1 + \mathbf{k})X(\mathbf{i} + \mathbf{j}), \\ \mu_{*(-\mathbf{i}-\mathbf{j})}(X) &= 0. \end{aligned}$$

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